Extended discrete KP hierarchy and its dispersionless limit *

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Abstract. We exhibit the dispersionless limit of the extended discrete KP hierarchy.

Keywords: discrete KP hierarchy, dispersionless limit

1. Introduction

There are many activities concerning with dispersionless hierarchies. The interest to these hierarchies comes, in particular, from string theory and related areas (see, for example [1], [2], [3]). Of course, one can also mention many other interesting themes like the study of the system of hydrodynamic type [4], the theory of conformal maps [5], [6] etc.

This letter is designed to present the extension of the dispersionless discrete KP hierarchy in a hope that this can be also useful in modern string physics or elsewhere. To this aim, we formulate the extended discrete KP (edKP) hierarchy in a suitable form in Section 2. In our opinion this representation is well background for performing dispersionless limit and for deriving Lax equations. This is doing in Section 3. In Appendix we display some evolution equations coded in corresponding Lax equations.

2. edKP hierarchy

We consider the space of analytic functions $\{f(s) : s \in \mathbb{R}\}$ whose domain of definition is restricted to $\mathbb{Z}$. We write $f = f(i) \in R$. On R the shift operator $\Lambda = \exp(\partial/\partial s)$ acts as $(\Lambda f)(i) = f(i+1)$.

Let us consider the space of pseudo-difference ($\triangle$DO) operators $\mathcal{D} = R[\Lambda, \Lambda^{-1})$ of the form

$$P = \sum_{j=-\infty}^{N} p_j \Lambda^j, \quad p_j \in R$$

* This work was supported by INTAS grant 2000-15 and RFBR grant 03-01-00102.

where $N \in \mathbb{Z}$ is treated as an order of P.

Phase-space of the discrete KP hierarchy can be identified with that of Zakharov-Shabat discrete dressing operators

$$\mathcal{M} = \{S = I + \sum_{k\geq 1} w_k \Lambda^{-k} : w_k \in R\}.$$

One considers the splitting $\mathcal{D} = \mathcal{D}_+^{(1)} \oplus \mathcal{D}_-^{(1)}$ with $\mathcal{D}_+^{(1)} \equiv < I, \Lambda, \Lambda^2, ... >$ and $\mathcal{D}_-^{(1)} \equiv < \Lambda^{-1}, \Lambda^{-2}, ... >$. The discrete KP hierarchy is given by Sato-Wilson

$$\frac{\partial S}{\partial t_p^{(1)}} = -\pi_-^{(1)}(Q^p)S = \pi_+^{(1)}(Q^p)S - S\Lambda^p$$

or equivalently by Lax equations

$$\frac{\partial Q}{\partial t_p^{(1)}} = [\pi_+^{(1)}(Q^p), Q] = [Q, \pi_-^{(1)}(Q^p)]$$

where the symbols $\pi_+^{(1)}$ and $\pi_-^{(1)}$ stand for projections of any $\triangle$DO on the space $\mathcal{D}_+^{(1)}$ and $\mathcal{D}_-^{(1)}$, respectively. The discrete Lax operator is defined as usual:

$$Q = S\Lambda S^{-1} = \Lambda + \sum_{k\geq 1} q_k \Lambda^{1-k}.$$

Fixing any point $\mathcal{M}$ and integer $n \geq 2$, we consider the splitting $\mathcal{D} = \mathcal{D}_+^{(n)} \oplus \mathcal{D}_-^{(n)}$ with

$$\mathcal{D}_+^{(n)} \equiv < I, \Lambda^n Q^{1-n}, \Lambda^{2n} Q^{2(1-n)}, ... >$$

and

$$\mathcal{D}_-^{(n)} \equiv < \Lambda^{-n} Q^{n-1}, \Lambda^{-2n} Q^{2(n-1)}, ... > .$$

One can see that this splitting (for $n \geq 2$) essentially depends on the point of the phase-space $\mathcal{M}$. For any $P \in \mathcal{D}$ one can write

$$P = p_N \Lambda^{Nn} Q^{N(1-n)} + p'_{N-1} \Lambda^{(N-1)n} Q^{(N-1)(1-n)} + ...$$

This representation is correctly defined since $\Lambda^{kn} Q^{k(1-n)}$ is a $\triangle$DO of an order k and therefore one can step-by-step calculate the coefficients $p'_{N-1}, p'_{N-2}, ...$ as polynomials of p_k and q_k.

We define edKP hierarchy by

$$\frac{\partial S}{\partial t_p^{(n)}} = -\pi_-^{(n)}(Q^p)S = \pi_+^{(n)}(Q^p)S - S\Lambda^p$$

or equivalently by

$$\frac{\partial Q}{\partial t_p^{(n)}} = [\pi_+^{(n)}(Q^p), Q] = [Q, \pi_-^{(n)}(Q^p)] \tag{1}$$

with $\pi_+^{(1)}$ and $\pi_-^{(1)}$ being projections on corresponding spaces. One can easy prove that these equations are correctly defined making of use standard reasonings.

Let $\chi(z) \equiv (z^i)_{i\in\mathbb{Z}}$ where z can be interpreted as a (formal) spectral parameter. Define $w(z) = S\chi(z)$ and formal Baker-Akhiezer function $\Psi(t,z) = w(z)e^{\xi(t,z)}$, where $\xi(t,z) \equiv \sum_{n,p=1}^{\infty} t_p^{(n)} z^p$.

It is common of knowledge that Lax equations like (1) can be understood as consistency condition of the linear system

$$Q\Psi = z\Psi, \;\; \frac{\partial \Psi}{\partial t_p^{(n)}} = \pi_+^{(n)}(Q^p)\Psi.$$

In what follows we are going to show that this system can be cast in the form

$$Q_{(n)}\Psi = z\Psi, \;\; z^{p(n-1)}\frac{\partial \Psi}{\partial t_p^{(n)}} = (Q_{(n)}^{pn})_+\Psi \tag{2}$$

with $Q_{(n)} \equiv S_{(n)}\Lambda S_{(n)}^{-1}$ where $S = I + \sum_{k\geq 1} z^{k(n-1)} w_k \Lambda^{-kn}$. The symbol $+$ denotes taking only nonnegative powers of Λ.

Multiplication $\chi(z)$ by z is equivalent to the action of the shift operator. Therefore one can easy to check that as a result of action $S_{(n)}$ to $\chi(z)$ for any integer $n \geq 1$ is $w(z)$ and from this we obtain that $Q_{(n)}w(z) = zw(z)$ or $Q_{(n)}\Psi = z\Psi$. To succeed it is useful to define, on $\mathcal{M}$, the functions $q_k^{(n,r)}$ through the relation

$$Q_{(n)}^r = S_{(n)}\Lambda^r S_{(n)}^{-1} = \Lambda^r + \sum_{k\geq 1} z^{k(n-1)} q_k^{(n,r)} \Lambda^{r-kn}.$$

By definition, we have $Q_{(n)}^r\Psi = z^r\Psi$.

Let us prove that for an arbitrary pair of integers $n, n' \geq 1$ the relation

$$Q_{(n')}^r = \Lambda^r + \sum_{k\geq 1} z^{k(n'-1)} q_k^{(n,r)} \Lambda^{r-kn} Q_{(n')}^{k(n-n')} \tag{3}$$

is valid.

Firstly we have

$$Q_{(n')}^r\Psi \;=\; \left\{ \Lambda^r + \sum_{k\geq 1} z^{k(n'-1)} q_k^{(n,r)} \Lambda^{r-kn} Q_{(n')}^{k(n-n')} \right\} \Psi$$

$$= \left\{ \Lambda^r + \sum_{k\geq 1} z^{k(n-1)} q_k^{(n,r)} \Lambda^{r-kn} \right\} \Psi$$

$$= Q_{(n)}^r \Psi = z^r \Psi.$$

Moreover it is obvious that LHS and RHS of (3) are of the same form.

More explicitly one can rewrite (3) as

$$q_k^{(n',r)}(i) = q_k^{(n,r)}(i) + \sum_{m=1}^{k-1} q_m^{(n,r)}(i) q_{k-m}^{(n',m(n-n'))}(i+r-mn).$$

In particular $q_1^{(n',r)}(i) = q_1^{(n,r)}(i)$. Putting in (3) $n' = 1$, we obtain

$$Q^r = \Lambda^r + \sum_{k\geq 1} q_k^{(n,r)} \Lambda^{r-kn} Q^{k(n-1)}.$$

Let $r = pn$ with some integer $p \geq 1$, then

$$Q^{pn} = \Lambda^{pn} + \sum_{k\geq 1} q_k^{(n,pn)} \Lambda^{(p-k)n} Q^{k(n-1)}.$$

Multiplying LHS and RHS of the latter by $Q^{p(1-n)}$, we get

$$Q^{pn} \cdot Q^{p(1-n)} = Q^p = \Lambda^{pn} Q^{p(1-n)} + \sum_{k\geq 1} q_k^{(n,pn)} \Lambda^{(p-k)n} Q^{(p-k)(1-n)}$$

The latter formula gives suitable form to split Q^p on 'positive' and 'negative' parts. In particular

$$\pi_+^{(n)}(Q^p) = \Lambda^{pn} Q^{p(1-n)} + \sum_{k=1}^{p} q_k^{(n,pn)} \Lambda^{(p-k)n} Q^{(p-k)(1-n)}.$$

From this we have

$$\pi_+^{(n)}(Q^p)\Psi = z^{p(1-n)} \left\{ \Lambda^{pn} + \sum_{k=1}^{p} z^{k(n-1)} q_k^{(n,pn)} \Lambda^{(p-k)n} \right\} \Psi$$

$$\equiv z^{p(1-n)} (Q_{(n)}^{pn})_+ \Psi.$$

The latter gives the second equation in (2). The consistency condition of (2) is expressed in the form of the following Lax equations:

$$z^{p(n-1)} \frac{\partial Q_{(n)}}{\partial t_p^{(n)}} = [(Q_{(n)}^{pn})_+, Q_{(n)}]. \tag{4}$$

Let us spend some lines to give bibliographical remarks concerning (4). These equations, in fact, are equivalent to Kupershmidt's gap KP hierarchy (see, for example [7]). More exactly, he considers Lax operator with anti-normal ordering

$$L = \Lambda + \Lambda^{1-\Gamma} \circ q_0 + \Lambda^{1-2\Gamma} \circ q_1 + ..., \quad \Gamma \geq 1$$

and corresponding Lax equations

$$\frac{\partial L}{\partial t_p} = [L_+^{p\Gamma}, L]. \tag{5}$$

In Ref. [7] the problem of integrable discretization of the flows given by (5) is solved.

In Refs. [8] and [9] we exhibit two-parametric class of invariant submanifolds $\mathcal{S}_l^n$ of the Darboux-KP chain [10]. Moreover, we showed that double intersections $\mathcal{S}_{n,r,l} = \mathcal{S}_0^n \cap \mathcal{S}_{l-1}^{ln-r}$ lead to a broad class of integrable lattices over finite number of fields (see also [11], [12] and references therein). All these systems share Lax representation of the form

$$z^{p(n-1)} \frac{\partial Q_{(n)}^r}{\partial t_p^{(n)}} = [(Q_{(n)}^{pn})_+, Q_{(n)}^r].$$

with restricted Lax operator

$$Q_{(n)}^r = \Lambda^r + \sum_{k=1}^{l} z^{k(n-1)} q_k^{(n,r)} \Lambda^{r-kn}.$$

It is important to note that r in this case cannot be treated as a power of $Q_{(n)}$.

The equations (1) have its advantage that they clearly show that the flows labeled by integers $n \geq 2$ are defined on the same phase-space as for the customary discrete KP hierarchy and in our opinion this representation is more convenient for performing dispersionless limit.

3. Dispersionless edKP hierarchy

Dispersionless limit of the discrete KP hierarchy or more generally of the Toda lattice one [13], [14] is performed by replacing

$$\frac{\partial}{\partial t_p} \to \hbar \frac{\partial}{\partial t_p}, \quad e^{\partial_s} \to e^{\hbar \partial_s}, \quad q_k \to \hbar^{-1} q_k$$

and tending $\hbar$ to zero. Then ∂_s is replaced by formal parameter p[1] and spectral problem turn into

$$z = e^p + q_1 + q_2 e^{-p} + q_3 e^{-2p} + ...$$

In turn, the dispersionless discrete KP hierarchy is defined by corresponding Lax equations

$$\frac{\partial z}{\partial t_p^{(1)}} = \{\pi_+^{(1)}(z^p), z\} = \{z, \pi_-^{(1)}(z^p)\}.$$

The commutator $[\cdot,\cdot]$ on $\mathcal{D}$ in the dispersionless limit is replaced by Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial p} = k\left(\frac{\partial f}{\partial k}\frac{\partial g}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial k}\right), \quad k = e^p.$$

Symbols $\pi_+^{(1)}$ and $\pi_-^{(1)}$ in this case stand for projections on the spaces $\tilde{\mathcal{D}}_+^{(1)} \equiv < 1, k, k^2, ... >$ and $\tilde{\mathcal{D}}_-^{(1)} \equiv < k^{-1}, k^{-2}, ... >$ which split on 'positive' and 'negative' parts the space $\tilde{\mathcal{D}}$ of formal Laurent series of the form

$$\tilde{P} = \sum_{j=-\infty}^{N} \tilde{p}_j k^j$$

with some coefficients $\tilde{p}_k$ being analytic functions of the variable s.

We can immediately to write down the equations which are a result of dispersionless limit applied to edKP hierarchy in the form of (1). We have

$$\frac{\partial z}{\partial t_p^{(n)}} = \{\pi_+^{(n)}(z^p), z\} = \{z, \pi_-^{(n)}(z^p)]. \tag{6}$$

Now the symbols $\pi_+^{(n)}$ and $\pi_-^{(n)}$ are used to denote projections on the spaces

$$\tilde{\mathcal{D}}_+^{(n)} \equiv < 1, z^{1-n}k^n, z^{2(1-n)}k^{2n}, ... >$$

and

$$\tilde{\mathcal{D}}_-^{(n)} \equiv < z^{n-1}k^{-n}, z^{2(n-1)}k^{-2n}, ... >,$$

respectively.

We aware that the (dispersionless) discrete KP hierarchy is only a part of the (dispersionless) Toda one and we are going in future to discuss extended version of the Toda lattice hierarchy.

[1] Note that this p has nothing to do with integer p labeled evolution parameters

Appendix

a. Let us give here explicit form of some functions $q_k^{(n,r)}$. We calculate first three of them to wit

$$q_1^{(n,r)}(i) = w_1(i) - w_1(i+r) = q_k^{(r)}(i),$$

$$\begin{aligned} q_2^{(n,r)}(i) &= w_2(i) - w_2(i+r) + w_1(i+r-n)(w_1(i+r) - w_1(i)) \\ &= q_2^{(r)}(i) - q_1^{(r)}(i) q_1^{(n-1)}(i+r-n), \\ q_3^{(n,r)}(i) &= w_3(i) - w_3(i+r) + w_1(i+r-2n)(w_2(i+r) - w_2(i)) \\ &\quad + w_2(i+r-n)(w_1(i+r) - w_1(i)) \\ &\quad + w_1(i+r-2n) w_1(i+r-n)(w_1(i) - w_1(i+r)) \\ &= q_3^{(r)}(i) - q_1^{(r)}(i) q_2^{(n-1)}(i+r-n) - q_2^{(r)}(i) q_1^{(2(n-1))}(i+r-2n) \\ &\quad + q_1^{(r)}(i) q_1^{(n-1)}(i+r-n) q_1^{(2(n-1))}(i+r-2n). \end{aligned}$$

Here $q_k^{(r)}$'s are the coefficients of r-th power of Q, i.e.

$$Q^r = \Lambda^r + q_1^{(r)} + q_2^{(r)} \Lambda^{-1} + \ldots$$

b. Let us exhibit some examples of equations of motion coded in (6) for $n = 2$. We have the following:

$$\frac{\partial q_1}{\partial t_1^{(2)}} = 2(q_2' - q_1 q_1'), \quad \frac{\partial q_2}{\partial t_1^{(2)}} = 2(q_3' - q_1 q_2' + q_1^2 q_1'),$$

$$\frac{\partial q_3}{\partial t_1^{(2)}} = 2(q_4' - q_1 q_3' + (q_1^2 - q_2) q_2' - (q_1^3 - 2 q_1 q_2 - q_3) q_1'), \ldots$$

$$\frac{\partial q_1}{\partial t_2^{(2)}} = 4 q_3', \quad \frac{\partial q_2}{\partial t_2^{(2)}} = 4(q_4' + (q_1^2 - q_2) q_2' - (q_1^3 - 2 q_1 q_2 - q_3) q_1'), \ldots$$

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